Chaining Syllogism Applied to Fuzzy IF-THEN Rules and Rule Bases

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Abstract. This paper presents a new approach to show the validity of the chaining syllogism for fuzzy IF-THEN rules and rule bases. Based on this approach new conditions are given on which the deduction scheme holds.

1 Introduction

In the two-valued propositional calculus the chaining syllogism

$$(p \to q) \wedge (q \to r) \to (p \to r),$$

where p, q and r are propositional variables, is a tautology. The validity of this deduction scheme applied to fuzzy IF-THEN rules and rule bases is studied in this paper, a problem investigated e.g. in [3,6,11] and recently in [2,5,8]. The chaining syllogism provides a way to combine blocks of fuzzy IF-THEN rules e.g. to reduce the complexity of a fuzzy controller or expert system.

Let U be an arbitrary non-empty set called universe. Fuzzy sets on U are mappings $F : U \to [0,1]$. The power set of $[0,1]$ is denoted by $\mathcal{P}([0,1])$ and the set of all fuzzy sets on U by $\mathcal{F}(U)$. Let $F, G, H \in \mathcal{F}(U)$. The range of a fuzzy set is defined as $RANGE(G) =_{def} \{G(x) | x \in U\}$.

2 Chainability of single rules

We consider rules of the form IF F THEN G. The case of multiple antecedents or several universes can be covered by using boolean expressions of fuzzy sets resp. the construction of cylindrical extension.

Definition 2.1
$\Im = [\pi, \kappa, Q]$ *is an interpretation of an IF-THEN rule*
$=_{def}$ *1.* $\pi, \kappa : [0,1]^2 \to [0,1]$
2. $Q : \mathcal{P}([0,1]) \to [0,1]$.

For a given interpretation $\Im$ the "implication function" function π interprets the rule IF F THEN G by defining the binary fuzzy relation

$$\Theta_\pi^{F,G}(x,y) =_{def} \pi(F(x), G(y)).$$

For an arbitrary $F' \in \mathcal{F}(U)$ the image $G' \in \mathcal{F}(U)$ is inferred by a generalised form of the compositional rule of inference (see [10]) as follows

$$G'(y) =_{def} (F' \circ_{Q,\kappa} \Theta_{\pi}^{F,G})(y),$$

where the functions κ and Q define the Q-κ-composition $\circ_{Q,\kappa}$ of fuzzy relations, e.g.

$$(F' \circ_{Q,\kappa} \Theta_{\pi}^{F,G})(y) = Q\{\kappa(F'(x), \Theta_{\pi}^{F,G}(x,y)) | x \in U\}.$$

Two rules IF F THEN G and IF G THEN H are called chainable iff the deduction scheme

$$\begin{array}{c} \text{IF } F \text{ THEN } G \\ \text{IF } G \text{ THEN } H \\ \hline \text{IF } F \text{ THEN } H \end{array}$$

is valid. An intuitive understanding of the validity of this "chaining scheme" leads to the following definition.

Definition 2.2
Two fuzzy rules IF F THEN G and IF G THEN H are called chainable with respect to $\Im$ and $\mathcal{F}' \subseteq \mathcal{F}(U)$ iff

$$\forall F'(F' \in \mathcal{F}' \to (F' \circ_{Q,\kappa} \Theta_{\pi}^{F,G}) \circ_{Q,\kappa} \Theta_{\pi}^{G,H} = F' \circ_{Q,\kappa} \Theta_{\pi}^{F,H}).$$

On which conditions are two rules chainable? The main problem is to show the chainability property for every input fuzzy set $F' \in \mathcal{F}'$. This leads to the following new approach.

Definition 2.3
Two fuzzy rules IF F THEN G and IF G THEN H are called "loosely" chainable with respect to $\Im$ iff

$$\Theta_{\pi}^{F,G} \circ_{Q,\kappa} \Theta_{\pi}^{G,H} = \Theta_{\pi}^{F,H}.$$

We can split the problem of chainability into two separate questions

① On which conditions does "loose" chainability imply chainability? The following section describes some sufficient conditions.
② On which conditions are rules "loosely" chainable? Examples are given in the third section.

Different answers to ① and ② can be combined and yield to various "chainability theorems".

3 Chainability of single rules implied by "loose" chainability

There is a close relation between the associativity of the Q-κ-composition, chainability and "loose" chainability.

Lemma 3.1
If the Q-κ-composition is associative then two fuzzy rules IF F THEN G and IF G THEN H are chainable if they are "loosely" chainable.

To characterize an important class of associative compositions we need the following property.

Definition 3.1
A mapping $\lambda : [0,1] \to [0,1]$ is lower-semicontinuous

$$=_{def} \forall x_0 (x_0 \in [0,1] \to \forall \epsilon (\epsilon > 0 \to \exists \delta (\delta > 0 \land \forall x (|x - x_0| < \delta \to \lambda(x_0) - \lambda(x) < \epsilon)))).$$

Lemma 3.2
Every nondecreasing mapping $\lambda : [0,1] \to [0,1]$ is lower-semicontinuous iff it is left continuous.

Based on [1,4] one can prove

Lemma 3.3
If 1. κ is nondecreasing and associative
2. every partial mapping of κ is lower-semicontinuous
3. $Q = Sup$
then the Q-κ-composition is associative.

Lemma 3.1 and 3.3 lead to the following theorem.

Theorem 3.4
If 1. κ nondecreasing and associative
2. every partial mapping of κ is lower-semicontinuous
3. $Q = Sup$
then two fuzzy rules IF F THEN G and IF G THEN H are chainable if they are "loosely" chainable.

If a partial mapping of κ is not lower-semicontinuous one can restrict the arguments of Q to finite sets to achieve the associativity of the Q-κ-composition.

Lemma 3.5
If 1. κ is nondecreasing and associative
2. $Q = Sup$
3. the arguments of Q are always finite sets
then the Q-κ-composition is associative.

With Lemma 3.1 we get

Theorem 3.6
If 1. κ *is nondecreasing and associative*
2. $Q = Sup$
3. for every $F' \in \mathcal{F}'$ *the union* $RANGE(F') \cup RANGE(F) \cup RANGE(G)$ *a finite set*

then two fuzzy rules IF F *THEN* G *and IF* G *THEN* H *are chainable with respect to* $\mathcal{F}'$ *if they are "loosely" chainable.*

If U is a finite set, then assumption 3 of the previous theorem is fulfilled.

For many (fuzzy control) applications the input can be restricted to "crisp" fuzzy sets of the form

$$F'_{x_0}(x) = \begin{cases} 1 \text{ ; if } x = x_0 \\ 0 \text{ ; else} \end{cases}$$

where $x, x_0 \in U$. ¿From [8] we quote the following lemma.

Lemma 3.7
If 1. $\forall r (r \in [0,1] \rightarrow \kappa(0,r) = 0 \wedge \kappa(1,r) = r)$
2. $\forall r (r \in [0,1] \rightarrow Q\{0,r\} = r)$

then for every $x_0, y \in U$

$$(F'_{x_0} \circ_{Q,\kappa} \Theta_\pi^{F,G})(y) = \Theta_\pi^{F,G}(x_0, y).$$

This leads to the following theorem, where the associativity of the Q-κ-composition is not necessarily assumed.

Theorem 3.8
If 1. $\forall r (r \in [0,1] \rightarrow \kappa(0,r) = 0 \wedge \kappa(1,r) = r)$
2. $\forall r (r \in [0,1] \rightarrow Q\{0,r\} = r)$

then two fuzzy rules IF F *THEN* G *and IF* G *THEN* H *are chainable with respect to "crisp" inputs if they are "loosely" chainable.*

4 "Loose" chainability of single rules

We give three examples for the "loose" chainability of single IF-THEN rules, which merge with the theorems of the previous section into different "chainability theorems".

Theorem 4.1
If 1. $\forall r \forall t (r, t \in [0,1] \rightarrow Q\{\kappa(\pi(r,s), \pi(s,t)) | s \in I\} = \pi(r,t))$
2. $RANGE(G) = I$

then

$$\Theta_\pi^{F,H} = \Theta_\pi^{F,G} \circ_{Q,\kappa} \Theta_\pi^{G,H}.$$

In [2] property 1 of Theo. 4.1 with $I = [0, 1]$ and $Q = Sup$ is investigated for several "implication functions" and t-norms. These results can also serve as sufficient conditions for the transitivity of the "implication functions" with respect to the t-norms.

Definition 4.1
$\pi : [0, 1]^2 \to [0, 1]$ *is called* κ*-transitive*
$=_{def} \forall r \forall s \forall t (r, s, t \in [0, 1] \to \kappa(\pi(r, s), \pi(s, t)) \leq \pi(r, t))$.

Theorem 4.2
If 1. $\forall r (r \in [0, 1] \to \pi(1, r) = r)$
2. $\forall r \forall s (r, s \in [0, 1] \to \pi(r, s) \leq \kappa(\pi(r, 1), s))$
3. π *is* κ*-transitive*
4. $Q = Sup$
5. G *is a normal fuzzy set on* U*, i.e.* $\exists y (y \in U \wedge G(y) = 1)$
then

$$\Theta_\pi^{F,H} = \Theta_\pi^{F,G} \circ_{Q,\kappa} \Theta_\pi^{G,H}.$$

If π is a t-norm and $\kappa = \pi$ assumptions 1, 2 and 3 of Theo. 4.2 are fulfilled. In fact if $Q = Sup$ and π is a t-norm then $\kappa = \pi$ is a necessary condition for the "loose" chainability.

Theorem 4.3
If 1. π *is reflexive, i.e.* $\forall r (r \in [0, 1] \to \pi(r, r) = 1)$
2. $\forall r (r \in [0, 1] \to \kappa(1, r) = r)$
3. π *is* κ*-transitive*
4. $Q = Sup$
5. $\forall x (x \in U \to \exists y (y \in U \wedge F(x) = G(y)))$
then

$$\Theta_\pi^{F,H} = \Theta_\pi^{F,G} \circ_{Q,\kappa} \Theta_\pi^{G,H}.$$

If κ is a lower-semicontinuous t-norm and π is the r-implication generated by κ, i.e. for all $r, s \in [0, 1]$

$$\pi(r, s) = Sup\{t | \kappa(r, t) \leq s \text{ and } t \in [0, 1]\},$$

then the assumptions 1-3 of Theo. 4.3 are fulfilled (see [4,5]).

5 Chainability of rule bases

For a positive natural number n, let $F_1, \ldots, F_n$, $G_1, \ldots, G_n$, $H_1, \ldots, H_n \in \mathcal{F}(U)$. We consider fuzzy IF-THEN rule bases of the form

$$RB_1 : \begin{array}{c} \text{IF } F_1 \text{ THEN } G_1 \\ \vdots \\ \text{IF } F_n \text{ THEN } G_n \end{array}$$

and introduce the abbreviations $\mathfrak{F} =_{def} [F_1, \ldots, F_n]$, $\mathfrak{G} =_{def} [G_1, \ldots, G_n]$ and RB_1 : IF $\mathfrak{F}$ THEN $\mathfrak{G}$.

Definition 5.1
$\mathfrak{I} = [\pi_1, \ldots, \pi_n, \kappa_0, \kappa_1, \ldots, \kappa_n, Q_0, Q_1 \ldots, Q_n, \alpha, \beta]$ *is called an interpretation of an IF-THEN rule base*
$=_{def}$ *1.* $\pi_1, \ldots, \pi_n, \kappa_0, \kappa_1, \ldots, \kappa_n : [0,1]^2 \to [0,1]$
2. $Q_0, Q_1 \ldots, Q_n : \mathcal{P}([0,1]) \to [0,1]$
3. $\alpha, \beta : [0,1]^n \to [0,1]$

For a given interpretation $\mathfrak{I}$ we define the "superrelation" $\Theta_{\mathfrak{I}}^{\mathfrak{F},\mathfrak{G}}$ and the operators $FATI_{\mathfrak{I}}^{\mathfrak{F},\mathfrak{G}}$ and $FITA_{\mathfrak{I}}^{\mathfrak{F},\mathfrak{G}}$ as follows.

Definition 5.2
1. $\Theta_{\mathfrak{I}}^{\mathfrak{F},\mathfrak{G}}(x,y) =_{def} \alpha(\pi_1(F_1(x), G_1(y)), \ldots, \pi_n(F_n(x), G_n(y)))$

2. $FATI_{\mathfrak{I}}^{\mathfrak{F},\mathfrak{G}}(F')(y) =_{def} (F' \circ_{\kappa_0, Q_0} \Theta_{\mathfrak{I}}^{\mathfrak{F},\mathfrak{G}})(y) =$
$Q_0\{\kappa_0(F'(x), \alpha(\pi_1(F_1(x), G_1(y)), \ldots, \pi_n(F_n(x), G_n(y))))|x \in U\}$

3. $FITA_{\mathfrak{I}}^{\mathfrak{F},\mathfrak{G}}(F')(y) =_{def} \beta\{Q_1\{\kappa_1(F'(x), \pi_1(F_1(x), G_1(y)))|x \in U\}, \ldots,$
$Q_n\{\kappa_n(F'(x), \pi_n(F_n(x), G_n(y)))|x \in U\}\}$

We need two further rule bases

$$RB_2 : \begin{array}{c} \text{IF } G_1 \text{ THEN } H_1 \\ \vdots \\ \text{IF } G_n \text{ THEN } H_n \end{array} \quad \text{and } RB_3 : \begin{array}{c} \text{IF } F_1 \text{ THEN } H_1 \\ \vdots \\ \text{IF } F_n \text{ THEN } H_n \end{array}$$

or for shortness RB_2 : IF $\mathfrak{G}$ THEN $\mathfrak{H}$ and RB_3 : IF $\mathfrak{F}$ THEN $\mathfrak{H}$ with $\mathfrak{H} =_{def} [H_1, \ldots, H_n]$.

By generalizing Def. 2.2 we obtain two versions of chainability of fuzzy rule bases.

Definition 5.3
1. *Two rule bases IF $\mathfrak{F}$ THEN $\mathfrak{G}$ and IF $\mathfrak{G}$ THEN $\mathfrak{H}$ interpreted by the principle FATI are called chainable with respect to $\mathfrak{I}$ and $\mathcal{F}'$, iff*

$$\forall F'(F' \in \mathcal{F}' \to FATI_{\mathfrak{I}}^{\mathfrak{G},\mathfrak{H}}(FATI_{\mathfrak{I}}^{\mathfrak{F},\mathfrak{G}}(F')) = FATI_{\mathfrak{I}}^{\mathfrak{F},\mathfrak{H}}(F')).$$

2. *Two rule bases IF $\mathfrak{F}$ THEN $\mathfrak{G}$ and IF $\mathfrak{G}$ THEN $\mathfrak{H}$ interpreted by the principle FITA are called chainable with respect to $\mathfrak{I}$ and $\mathcal{F}'$, iff*

$$\forall F'(F' \in \mathcal{F}' \to FITA_{\mathfrak{I}}^{\mathfrak{G},\mathfrak{H}}(FITA_{\mathfrak{I}}^{\mathfrak{F},\mathfrak{G}}(F')) = FITA_{\mathfrak{I}}^{\mathfrak{F},\mathfrak{H}}(F')).$$

The considerations of the previous sections also hold for the chaining of IF-THEN rule bases interpreted by the principle FATI, where the rules of each rule base are aggregated to a single "superrelation".

Definition 5.4
Two rule bases IF $\mathfrak{F}$ THEN $\mathfrak{G}$ and IF $\mathfrak{G}$ THEN $\mathfrak{H}$ are called "loosely" chainable chainable with respect to $\mathfrak{I}$, iff

$$\Theta_{\mathfrak{I}}^{\mathfrak{F},\mathfrak{G}} \circ_{\kappa_0,Q_0} \Theta_{\mathfrak{I}}^{\mathfrak{G},\mathfrak{H}} = \Theta_{\mathfrak{I}}^{\mathfrak{F},\mathfrak{H}}.$$

Again we split the problem of chainability into two questions

① On which conditions does "loose" chainability imply chainability of rule bases interpreted by the principle FATI? Answers are given in the next section.
② On which conditions are rule bases "loosely" chainable? Examples are given in Sect. 7.

The results also hold for FITA, if the principles FATI and FITA coincide. The equivalence of the principles FATI and FITA is investigated in [7,10].

6 Chainability of rule bases interpreted by the principle FATI implied by "loose" chainability

The basic ideas are the same as in Sect. 3.

Theorem 6.1
If 1. κ_0 is nondecreasing and associative
2. every partial mapping of κ_0 is lower-semicontinuous
3. $Q_0 = Sup$
then two rule bases IF $\mathfrak{F}$ THEN $\mathfrak{G}$ and IF $\mathfrak{G}$ THEN $\mathfrak{H}$ interpreted by the principle FATI are chainable if they are "loosely" chainable.

Theorem 6.2
If 1. κ_0 is nondecreasing and associative
2. $Q_0 = Sup$
3. for every $F' \in \mathcal{F}'$ the union $RANGE(F') \cup RANGE(F_1) \cup \cdots \cup RANGE(F_n) \cup RANGE(G_1) \cup \cdots \cup RANGE(G_n)$ is a finite set
then two rule bases IF $\mathfrak{F}$ THEN $\mathfrak{G}$ and IF $\mathfrak{G}$ THEN $\mathfrak{H}$ interpreted by the principle FATI are chainable with respect to $\mathcal{F}$ if they are "loosely" chainable.

If U is a finite set, then assumption 3 of the previous theorem is fulfilled.

¿From [8] we quote the next lemma.

Lemma 6.3
If 1. $\forall r(r \in [0,1] \to \kappa_0(0,r) = 0 \wedge \kappa_0(1,r) = r)$
2. $\forall r(r \in [0,1] \to Q_0\{0,r\} = r)$
then for every $x_0, y \in U$

$$FATI_{\mathfrak{I}}^{\mathfrak{G},\mathfrak{H}}(F'_{x_0})(y) = \alpha(\pi_1(F_1(x_0), G_1(y)), \ldots, \pi_n(F_n(x_0), G_n(y)))$$

Theorem 6.4
If 1. $\forall r(r \in [0,1] \to \kappa_0(0,r) = 0 \wedge \kappa_0(1,r) = r)$
2. $\forall r(r \in [0,1] \to Q_0\{0,r\} = r)$

then two rule bases IF $\mathfrak{F}$ THEN $\mathfrak{G}$ and IF $\mathfrak{G}$ THEN $\mathfrak{H}$ interpreted by the principle FATI are chainable with respect to "crisp" inputs if they are "loosely" chainable.

7 "Loose" chainability of rule bases

Based on [8] we give two examples for the "loose" chainability of rule bases, which can be combined with the theorems of the previous section.

Definition 7.1

1. *α is monotone*
$=_{def} \forall r_1 \dots \forall r_n \forall s_1 \dots \forall s_n(r_1,\dots,r_n,s_1,\dots,s_n \in [0,1] \wedge$
$r_1 \le s_1 \wedge \dots \wedge r_n \le s_n \to \alpha(r_1,\dots,r_n) \le \alpha(s_1,\dots,s_n))$
2. *κ_0 is right-side strongly α-distributive*
$=_{def} \forall r_1 \dots \forall r_n \forall s_1 \dots \forall s_n(r_1,\dots,r_n,s_1,\dots,s_n \in [0,1]$
$\to \kappa_0(\alpha(r_1,\dots,r_n),\alpha(s_1,\dots,s_n)) \le \alpha(\kappa_0(r_1,s_1),\dots,\kappa_0(r_n,s_n)))$
3. *κ_0 is left-side weakly α-distributive*
$=_{def} \forall r_1 \dots \forall r_n \forall s_1 \dots \forall s_n(r_1,\dots,r_n,s_1,\dots,s_n \in [0,1]$
$\to \alpha(\kappa_0(r_1,s_1),\dots,\kappa_0(r_n,s_n))) \le \kappa_0(\alpha(r_1,\dots,r_n),\alpha(s_1,\dots,s_n)))$

Theorem 7.1
If 1. *for every $i \in \{1,\dots,n\}$, π_i is reflexive*
2. *for every $i \in \{1,\dots,n\}$, π_i is κ_0-transitive*
3. $\forall r(r \in [0,1] \to \kappa_0(1,r) = r)$
4. $Q_0 = Sup$
5. *α is monotone*
6. *κ_0 is right-side strongly α-distributive*
7. (a) *κ_0 is left-side weakly α-distributive* ***or***
(b) $\alpha(1,\dots,1) = 1$
8. $\forall x(x \in U \to \exists y(y \in U \wedge F_1(x) = G_1(y) \wedge \dots \wedge F_n(x) = G_n(y)))$,

then
$$\Theta_{\mathfrak{I}}^{\mathfrak{F},\mathfrak{G}} \circ_{\kappa_0,Q_0} \Theta_{\mathfrak{I}}^{\mathfrak{G},\mathfrak{H}} = \Theta_{\mathfrak{I}}^{\mathfrak{F},\mathfrak{H}}.$$

If e.g. 1. $\forall r \forall s(r,s \in [0,1] \to \pi_1(r,s) = \dots = \pi_n(r,s) = min(1, 1-r+s))$
2. $\forall r \forall s(r,s \in [0,1] \to \kappa_0(r,s) = max(0, r+s-1))$
3. $Q = Sup$
4. $\forall r_1 \dots \forall r_n(r_1,\dots,r_n \in [0,1] \to$
$\alpha(r_1,\dots,r_n) = \kappa_0(r_1,\kappa_0(r_2,\dots\kappa_0(r_{n-1},r_n)\dots)))$

then the assumptions 1-6 and 7b of the previous theorem are fulfilled, hence the two fuzzy rule bases are "loosely" chainable if assumption 8 is satisfied.

Theorem 7.2

If 1. for every $i \in \{1, \ldots, n\}$, $\forall r(r \in [0,1] \to \pi_i(1,r) = r)$,

2. for every $i \in \{1, \ldots, n\}$, $\forall r \forall s(r, s \in [0,1] \to \pi_i(r,s) \leq \kappa_0(\pi_i(r,1), s))$

3. for every $i \in \{1, \ldots, n\}$, π_i *is* κ_0*-transitive*

4. $Q_0 = Sup$

5. α *is monotone*

6. κ_0 *is right-side strongly* α*-distributive*

7. κ_0 *is left-side weakly* α*-distributive*

8. $\exists y(y \in U \wedge G_1(y) = \cdots = G_n(y) = 1)$

then

$$\Theta_{\mathfrak{J}}^{\mathfrak{F},\mathfrak{G}} \circ_{\kappa_0, Q_0} \Theta_{\mathfrak{J}}^{\mathfrak{G},\mathfrak{H}} = \Theta_{\mathfrak{J}}^{\mathfrak{F},\mathfrak{H}}.$$

If e.g. 1. $\pi_1 = \cdots = \pi_n = \kappa_0 = \kappa_1 = \cdots = \kappa_n = min$

2. $Q_0, Q_1, \ldots, Q_n = Sup$

3. $\alpha = \beta = Min$

then the assumptions 1-7 of the previous theorem are satisfied and FITA and FATI coincide.

8 Conclusion

The proposed systematic approach combines new considerations with the results of former investigations and leads to various new conditions on which fuzzy rules and rule bases are chainable.

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