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The Chaining Syllogism in Fuzzy Logic

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Abstract

We analyze the validity of the chaining syllogism in fuzzy systems, i.e., whether two fuzzy rules IF F THEN G and IF G THEN H imply the rule IF F THEN H. Conditions are given on which this basic deduction scheme holds and various common implication operators are analyzed with respect to these conditions.

εἰ γάρ τό Α κατά παντός τοῦ Β καί τό Β κατά παντός τοῦ Γ,
ανάγκη τό Α κατά παντός τοῦ Γ κατηγορεῖσθαι
If A is predicated of all B, and B of all C,
A must necessarily be predicated of all C

The chaining syllogism according to ARISTOTLE's *Prior Analytics*

Keywords: chaining syllogism, implication operators, rule chaining

1 Introduction

In the two-valued propositional calculus the chaining syllogism

$$(p \to q) \wedge (q \to r) \to (p \to r) \ ,$$

where p, q, and r are propositional variables, is a tautology—in fuzzy logic it is generally not. The chaining rule is one of the most important deduction schemes from a theoretical and also practical point of view, e.g., it allows to combine fuzzy IF-THEN rules and thereby to reduce the complexity of fuzzy controllers or expert systems. The validity of the chaining syllogism in fuzzy logic has been studied in [5, 11, 22, 4, 7, 14, 15, 17, 8, 10]. In this article, we extend our previous investigations and prove the up to now most general conditions under which fuzzy rules are chainable. The chainability depends on the implication function based on which the IF-THEN rules are interpreted. We analyze the relevant properties, which are not only important for rule chaining, of several common implication functions.

The next section introduces the notation and basic definitions. In section 3 we present sufficient conditions for the chainability of fuzzy IF-THEN rules. Section 4 summarizes properties of various implication functions.

2 Preliminaries

We follow the notation of [12, 16, 9]. Let U be an arbitrary non-empty set called universe. Fuzzy sets on U are mappings $U \to [0,1]$ and n-ary fuzzy relations are mappings $U^n \to [0,1]$. The power set of $[0,1]$ is denoted by $\mathcal{P}([0,1])$ and the set of all fuzzy sets on U by $\mathcal{F}(U)$. Let $F, G, H \in \mathcal{F}(U)$ throughout this article. The range of a fuzzy set G is defined as $\mathrm{range}(G) := \{G(x)|x \in U\}$.

We consider rules of the form IF F THEN G. The case of multiple antecedents or several universes can be covered by using boolean expressions of fuzzy sets and the construction of cylindrical extensions, respectively. The semantics of a rule is described by its *interpretation*:

Definition 1. *$\Im = [\pi, \kappa, Q]$ with $\pi, \kappa : [0,1]^2 \to [0,1]$ and $Q : \mathcal{P}([0,1]) \to [0,1]$ is called an interpretation of an IF-THEN rule.*

For a given $\Im$ the *implication function* π interprets the rule IF F THEN G by defining the binary fuzzy relation

$$\Theta_\pi^{F,G}(x,y) := \pi(F(x), G(y)) \ .$$

For any $F' \in \mathcal{F}(U)$ the image $G' \in \mathcal{F}(U)$ is inferred by the generalized form of the compositional rule of inference ([21], cf. [16, 9]) as follows

$$G'(y) := (F' \circ_{Q,\kappa} \Theta_\pi^{F,G})(y) \ ,$$

where the functions κ and Q define the Q-κ-composition $\circ_{Q,\kappa}$ of fuzzy relations:

$$(F' \circ_{Q,\kappa} \Theta_\pi^{F,G})(y) = Q\{\kappa(F'(x), \Theta_\pi^{F,G}(x,y))|x \in U\} \ .$$

This way of looking at fuzzy rules is inspired by the generalized modus ponens

$$\begin{array}{l} \text{IF } F \text{ THEN } G \\ \quad F' \\ \hline \qquad\qquad G' \end{array} \ .$$

3 Chainability of fuzzy IF-THEN rules

Two rules IF F THEN G and IF G THEN H are called chainable iff the deduction rule

$$\frac{\begin{array}{c}\text{IF } F \text{ THEN } G\\ \text{IF } G \text{ THEN } H\end{array}}{\text{IF } F \text{ THEN } H}$$

is valid. The intuitive understanding of the validity of this chaining scheme leads to the following definition:

Definition 2. *Two fuzzy rules IF F THEN G and IF G THEN H are called chainable with respect to $\Im$ and $\mathcal{F}' \subseteq \mathcal{F}(U)$ iff*

$$\forall F' \in \mathcal{F}' : (F' \circ_{Q,\kappa} \Theta_\pi^{F,G}) \circ_{Q,\kappa} \Theta_\pi^{G,H} = F' \circ_{Q,\kappa} \Theta_\pi^{F,H} \ .$$

On which conditions are two rules chainable? The main task is to show the chainability property for every input fuzzy set $F' \in \mathcal{F}'$. In our approach, we decompose the question using the concept of *loose chainability*:

Definition 3. *Two fuzzy rules IF F THEN G and IF G THEN H are called loosely chainable with respect to $\Im$ iff*

$$\Theta_\pi^{F,G} \circ_{Q,\kappa} \Theta_\pi^{G,H} = \Theta_\pi^{F,H}.$$

It is important to note that loose chainability does not necessarily imply chainability as assumed in [4]. Now we can split the problem of chainability into two separate questions:

1. On which conditions does loose chainability imply chainability? In 3.1 some sufficient conditions are proven.
2. On which conditions are rules loosely chainable? Examples are given in 3.2.

Different answers to 1 and 2 can be combined and yield various chainability theorems.

3.1 Chainability implied by loose chainability

3.1.1 Associativity of the Q-κ-composition

The Q-κ-composition is called associative iff

$$(R_1 \circ_{Q,\kappa} R_2) \circ_{Q,\kappa} R_3 = R_1 \circ_{Q,\kappa} (R_2 \circ_{Q,\kappa} R_3)$$

for arbitrary (binary) fuzzy relations R_1, R_2, R_3. It is straightforward to show that there is a close relation between the associativity of the Q-κ-composition, chainability, and loose chainability:

Lemma 1. *If the Q-κ-composition is associative, then two fuzzy rules IF F THEN G and IF G THEN H are chainable if they are loosely chainable.*

Proof. When the Q-κ-composition is associative, $(F' \circ_{Q,\kappa} \Theta_\pi^{F,G}) \circ_{Q,\kappa} \Theta_\pi^{G,H} = F' \circ_{Q,\kappa} \Theta_\pi^{F,H}$ for all $F' \in \mathcal{F}(U)$ implies $F' \circ_{Q,\kappa} (\Theta_\pi^{F,G} \circ_{Q,\kappa} \Theta_\pi^{G,H}) = F' \circ_{Q,\kappa} \Theta_\pi^{G,H}$, which holds for all $F' \in \mathcal{F}(U)$ if $\Theta_\pi^{F,G} \circ_{Q,\kappa} \Theta_\pi^{G,H} = \Theta_\pi^{G,H}$. □

Thus, we ask the question which Q-κ-compositions are associative. To characterize an important class of associative compositions we need the concept of lower-semicontinuity.

Definition 4. *A mapping $\lambda : [0,1] \to [0,1]$ is lower-semicontinuous iff*
$\forall x_0 \in [0,1] : \forall \epsilon > 0 : \exists \delta > 0 : \forall x \in [0,1] : (|x - x_0| < \delta \to \lambda(x_0) - \lambda(x) < \epsilon)$.

It holds [6, Proposition 1.2]:

Lemma 2. *Every nondecreasing mapping $\lambda : [0,1] \to [0,1]$ is lower-semicontinuous iff it is left continuous.*

Now one can prove [6, Proposition 2.13]:

Theorem 1. *If*

1. $Q = \sup$,
2. *κ is nondecreasing and associative, and*
3. *every partial mapping of κ is lower-semicontinuous,*

then the Q-κ-composition is associative.

Lemma 1 and theorem 1 lead to the following corollary:

Corollary 1. *If*

1. *κ nondecreasing and associative,*
2. *every partial mapping of κ is lower-semicontinuous, and*
3. $Q = \sup$,

then two fuzzy rules IF F THEN G and IF G THEN H are chainable if they are loosely chainable.

3.1.2 Finiteness of ranges

If a partial mapping of κ is not lower-semicontinuous one can restrict the arguments of Q to finite sets to achieve associativity of the Q-κ-composition [6, Proposition 2.13]:

Theorem 2. *If*

1. *κ is nondecreasing and associative,*
2. *the arguments of Q are always finite sets, and*
3. *$Q = \sup$,*

then the Q-κ-composition is associative.

With lemma 1 we get:

Corollary 2. *If*

1. *κ is nondecreasing and associative,*
2. *$Q = \sup$, and*
3. *for every $F' \in \mathcal{F}'$ the union $\mathrm{range}(F') \cup \mathrm{range}(F) \cup \mathrm{range}(G)$ is a finite set,*

then two fuzzy rules IF F THEN G and IF G THEN H are chainable with respect to $\mathcal{F}'$ if they are loosely chainable.

If U is a finite set, which is no restriction if we consider implementations on digital computers, then assumption 3 of the previous statement is always fulfilled.

3.1.3 Singleton inputs

For many (e.g., control) applications the input can be restricted to singleton ("crisp") fuzzy sets of the form

$$F'_{x_0}(x) = \begin{cases} 1 \ ; \ \text{if } x = x_0 \\ 0 \ ; \ \text{otherwise} \end{cases} ,$$

where $x, x_0 \in U$. Based on [14], we have:

Lemma 3. *If*

1. *$\forall r \in [0,1] : \kappa(0,r) = 0 \wedge \kappa(1,r) = r$ (i.e., κ is a semi-conjunction) and*
2. *$\forall r \in [0,1] : Q\{0,r\} = r$ (i.e., Q is a semi-or-quantifier),*

then for every $x_0, y \in U$

$$(F'_{x_0} \circ_{Q,\kappa} \Theta_\pi^{F,G})(y) = \Theta_\pi^{F,G}(x_0, y) \ .$$

Proof. For all $x_0, y \in U$ we have

$$\begin{aligned}
(F'_{x_0} \circ_{Q,\kappa} \Theta_\pi^{F,G})(y) &= Q\{\kappa(F'_{x_0}(x), \Theta_\pi^{F,G}(x,y))|x \in U\} \\
&= Q\,\{\kappa(F'_{x_0}(x_0), \Theta_\pi^{F,G}(x_0,y))\} \cup \{\kappa(F'_{x_0}(x), \Theta_\pi^{F,G}(x,y))|x \in U \setminus \{x_0\}\} \\
&= Q\,\{\kappa(1, \Theta_\pi^{F,G}(x_0,y))\} \cup \{\kappa(0, \Theta_\pi^{F,G}(x,y))|x \in U \setminus \{x_0\}\} \\
&= Q\{\Theta_\pi^{F,G}(x_0,y), 0\} = \Theta_\pi^{F,G}(x_0,y) \ .
\end{aligned}$$

□

This leads to the following theorem, where the associativity of the Q-κ-composition is not necessarily assumed:

Theorem 3. *If*

1. $\forall r \in [0,1] : \kappa(0,r) = 0 \wedge \kappa(1,r) = r$ *and*
2. $\forall r \in [0,1] : Q\{0,r\} = r$,

then two fuzzy rules IF F THEN G and IF G THEN H are chainable with respect to singleton inputs if they are loosely chainable.

Proof. The two rules are chainable with respect to singleton inputs if for all $x_0, z \in U$

$$Q\{\kappa((F'_{x_0} \circ_{Q,\kappa} \Theta_\pi^{F,G})(y), \pi(G(y), H(z)))|y \in U\} = F'_{x_0} \circ_{Q,\kappa} \Theta_\pi^{F,H}(z) \ .$$

As the conditions of lemma 3 are fulfilled, this is equivalent to

$$Q\{\kappa(\pi(F(x_0), G(y)), \pi(G(y), H(z)))|y \in U\} = \pi(F(x_0), H(z)),$$

which corresponds to loose chainability by definition. □

3.2 Loose chainability

In this section, we give three conditions for loose chainability of single IF-THEN rules, which can be combined with the results of the previous section to different chainability theorems.

Obviously, it holds:

Theorem 4. *If*

1. $\forall r, t \in [0,1] : Q\{\kappa(\pi(r,s), \pi(s,t))|s \in I\} = \pi(r,t)$ *and*
2. $\mathrm{range}(G) = I$,

then

$$\Theta_\pi^{F,H} = \Theta_\pi^{F,G} \circ_{Q,\kappa} \Theta_\pi^{G,H} \ .$$

In [4] property 1 of theorem 4 with $I = [0,1]$ and $Q = \sup$ is investigated for several implication functions and t-norms. These results can also serve as sufficient conditions for the κ-transitivity of the implication functions with respect to the t-norms:

Definition 5. $\pi : [0,1]^2 \to [0,1]$ *is called* κ*-transitive iff* $\forall r,s,t \in [0,1] : \kappa(\pi(r,s),\pi(s,t)) \leq \pi(r,t)$.

Now we can prove:

Lemma 4. π *is* κ*-transitive iff*

$$\forall x,z \in U : \pi(F(x),H(z)) \geq \sup\{\kappa(\pi(F(x),G(y)),\pi(G(y),H(z)))|y \in U\} \ .$$

Proof. "$\leftarrow$": Let π be κ-transitive and assume

$$\exists x_0, z_0 \in U : \sup\{\kappa(\pi(F(x_0),G(y)),\pi(G(y),H(z_0)))|y \in U\} > \pi(F(x_0),H(z_0)) \ .$$

Hence, $\exists y_0 \in U : \kappa(\pi(F(x_0),G(y_0)),\pi(G(y_0),H(z_0))) > \pi(F(x_0),H(z_0))$, which contradicts the κ-transitivity of π.

"$\rightarrow$": Let $\forall x,z \in U : \sup\{\kappa(\pi(F(x),G(y)),\pi(G(y),H(z)))|y \in U\} \leq \pi(F(x),H(z))$. This implies $\forall x,y,z \in U : \kappa(\pi(F(x),G(y)),\pi(G(y),H(z))) \leq \pi(F(x),H(z))$. Assume that π is not κ-transitive, i.e., $\exists r_0,s_0,t_0 \in [0,1] : \kappa(\pi(r_0,s_0),\pi(s_0,t_0)) > \pi(r_0,t_0)$, then setting $F(x_0) = r_0$, $G(y_0) = s_0$, and $H(z_0) = t_0$ leads to a contradiction. □

Theorem 5. *If*

1. $\forall r \in [0,1] : \pi(1,r) = r$,
2. $\forall r,s \in [0,1] : \pi(r,s) \leq \kappa(\pi(r,1),s)$,
3. π *is* κ*-transitive,*
4. $Q = \sup$*, and*
5. G *is a normal fuzzy set on* U *(i.e.,* $\exists y' \in U : G(y') = 1$*),*

then

$$\Theta_\pi^{F,H} = \Theta_\pi^{F,G} \circ_{Q,\kappa} \Theta_\pi^{G,H} \ .$$

Proof. Because of lemma 4, it is only left to show that

$$\pi(F(x),H(z)) \leq Q\{\kappa(\pi(F(x),G(y)),\pi(G(y),H(z)))|y \in U\}$$

for all $x,z \in U$, i.e., that

$$\exists y \in U : \pi(F(x),H(z)) \leq \kappa(\pi(F(x),G(y)),\pi(G(y),H(z)))$$

for all $x, z \in U$, see [14]. We have an element $y' \in U$ such that $G(y') = 1$ and hence

$$\begin{aligned} \kappa(\pi(F(x), G(y')), \pi(G(y'), H(z))) &= \kappa(\pi(F(x), 1), \pi(1, H(z))) \\ &= \kappa(\pi(F(x), 1), 1) \\ &\geq \pi(F(x), H(z)) \ . \end{aligned}$$

□

If π is a t-norm and $\kappa = \pi$, then assumptions 1, 2, and 3 of theorem 5 are fulfilled. In fact, if $Q = \sup$ and π is a t-norm, then $\kappa = \pi$ is a *necessary* condition for the loose chainability:

Lemma 5. *If*

1. $\forall r \in [0, 1] : \pi(1, r) = r \wedge \pi(r, 1) = r$ *and*
2. $Q = \sup$,

then $\Theta_\pi^{F,H} = \Theta_\pi^{G,H} \circ_{Q,\kappa} \Theta_\pi^{F,G}$ *implies* $\kappa = \pi$.

Proof. Let $\Theta_\pi^{F,H} = \Theta_\pi^{G,H} \circ_{Q,\kappa} \Theta_\pi^{F,G}$ for all $F, G, H \in \mathcal{F}(U)$. Assume $\kappa \neq \pi$, i.e., $\exists s, t \in [0, 1] : \kappa(s, t) \neq \pi(s, t))$. Hence, we can find $u, v \in U$ and $F', H' \in \mathcal{F}(U)$ with $\kappa(F'(u), H'(v)) \neq \pi(F'(u), H'(v))$. Let $G' \in \mathcal{F}(u)$ be universal (i.e., $\forall y \in U : G'(y) = 1$). Then we have

$$\begin{aligned} (\Theta_\pi^{G',H'} \circ_{Q,\kappa} \Theta_\pi^{F',G'})(u, v) &= Q\{\kappa(\pi(F'(u), G'(y)), \pi(G'(y), H'(v)))|y \in U\} \\ &= Q\{\kappa(\pi(F'(u), 1), \pi(1, H'(v)))|y \in U\} \\ &= Q\{\kappa(F'(u), H'(v))|y \in U\} \\ &= \kappa(F'(u), H'(v)) = \Theta_\pi^{F',G'}(u, v) = \pi(F'(u), H'(v)) \ , \end{aligned}$$

which contradicts the assumption. □

The following theorem gives alternative conditions for loose chainability:

Theorem 6. *If*

1. π *is reflexive, i.e.,* $r \in [0, 1] : \pi(r, r) = 1$,
2. $\forall r \in [0, 1] : \kappa(1, r) = r$,
3. π *is* κ*-transitive,*
4. $Q = \sup$, *and*
5. $\forall x \in U : \exists y \in U : F(x) = G(y)$,

then

$$\Theta_\pi^{F,H} = \Theta_\pi^{F,G} \circ_{Q,\kappa} \Theta_\pi^{G,H} \ .$$

Proof. We need to show that for all $x, z \in U$

$$\exists y \in U : \pi(F(x), H(z)) \leq \kappa(\pi(F(x), G(y)), \pi(G(y), H(z))) \ ,$$

see the proof of Theorem 5. As $\exists y \in [0,1] : F(x) = G(y)$ it suffices to show that

$$\forall x, z \in U : \pi(F(x), H(z)) \leq \kappa(\pi(F(x), F(x)), \pi(F(x), H(z))) \ .$$

Because of prerequisites 1 and 3, it holds

$$\kappa(\pi(F(x), F(x)), \pi(F(x), H(z))) = \kappa(1, \pi(F(x), H(z))) = \pi(F(x), H(z))$$

and therefore also the previous two inequalities. □

If κ is a lower-semicontinuous t-norm and π is the r-implication generated by κ, i.e., for all $r, s \in [0,1]$ $\pi(r,s) = \sup\{t | \kappa(r,t) \leq s, t \in [0,1]\}$, then the conditions 1-3 of theorem 6 are fulfilled (see [6, 7]).

The question arises whether theorems 5 and 6 cover all cases of loose chainability. The answer is no:

Lemma 6. *There are interpretations that fulfill all conditions of theorem 4 with $I = [0,1]$, but neither the ones of theorem 5 nor 6.*

Proof. The interpretation $[\pi_{1e}, \min, \sup]$ does not fulfill the conditions of theorems 5 and 6, but leads to loose chainability. Let $r, s \in [0,1]$. In [19] the function π_{1e} is defined as

$$\pi_{1e}(r,s) := \begin{cases} 0 \ ; \ x < y \\ y \ ; \ \text{otherwise} \end{cases} \ .$$

The mapping π_{1e} is not reflexive, because $\pi_{1e}(r,r) = r$, i.e., usually not 1. Hence, condition 1 of theorem 6 is violated. Let $1 > r > s$, then we have $\pi_{1e}(r,s) = s$ and $\min(\pi_{1e}(r,1), s) = \min(0, s) = 0$, which implies that condition 2 of theorem 5 does not hold.

However, $[\pi_{1e}, \min, \sup]$ can fulfill the conditions of theorem 4 with $I = [0,1]$. It suffices to show that $\sup\{\min(\pi_{1e}(x,y), \pi_{1e}(y,z) | y \in [0,1]) = \pi_{1e}(x,z)$ for all $x, z \in [0,1]$. For $x < z$ we have

$$\sup\{\min(\pi_{1e}(x,y), \pi_{1e}(y,z) | y \in [0,1]) =$$

$$\max \left\{ \begin{array}{l} \sup\{\min(\pi_{1e}(x,y), \pi_{1e}(y,z)) | y \in [0,x]\}, \\ \sup\{\min(\pi_{1e}(x,y), \pi_{1e}(y,z)) | y \in]x,z[\}, \\ \sup\{\min(\pi_{1e}(x,y), \pi_{1e}(y,z)) | y \in [z,1]\} \end{array} \right\} =$$

$$\max \left\{ \begin{array}{l} \sup\{\min(y,0) | y \in [0,x]\}, \\ \sup\{\min(0,0) | y \in]x,z[\}, \\ \sup\{\min(0,z) | y \in [z,1]\} \end{array} \right\} = 0 = \pi_{1e}(x,z)$$

and for $x \geq z$

$$\sup\{\min(\pi_{1e}(x,y),\pi_{1e}(y,z)|y \in [0,1]) =$$

$$\max\left\{\begin{array}{l}\sup\{\min(\pi_{1e}(x,y),\pi_{1e}(y,z))|y \in [0,z[\},\\ \sup\{\min(\pi_{1e}(x,y),\pi_{1e}(y,z))|y \in [z,x]\},\\ \sup\{\min(\pi_{1e}(x,y),\pi_{1e}(y,z))|y \in]x,1]\}\end{array}\right\} =$$

$$\max\left\{\begin{array}{l}\sup\{\min(y,0)|y \in [0,z[\},\\ \sup\{\min(y,z)|y \in [z,x]\},\\ \sup\{\min(0,z)|y \in]x,1]\}\end{array}\right\} = z = \pi_{1e}(x,z) \ .$$

$\square$

4 Properties of common implication functions

In this section, we study relevant properties of several implication functions taken from the literature. We restrict our considerations to the functions in table 1, a superset of the ones discussed in [3, 5, 11, 1, 2, 18] (the names are also taken from these articles). Let $\pi : [0,1]^2 \rightarrow [0,1]$. We analyze the following properties:

$I_0:$ $\pi(1,0) = 0 \wedge \pi(0,1) = \pi(0,0) = \pi(0,1) = 1$

$I_1:$ $I_1^a : \pi(0,x) = 1$
$I_1^b : \pi(1,x) = x$
$I_1^c : \pi(x,1) = 1$

$I_2:$ the function is nonincreasing in the first and nondecreasing in the second argument, i.e, $x \leq y \rightarrow \pi(x,z) \geq \pi(y,z) \wedge \pi(z,x) \leq \pi(z,y)$

$I_3:$ $\pi(x,y) = \pi(\pi(y,0),\pi(x,0))$

$I_4:$ $\pi(x,\pi(y,z)) = \pi(y,\pi(x,z))$

$I_5:$ the function is reflexive, i.e., $\pi(x,x) = 1$

$I_6:$ $\pi(x,\pi(y,x)) = 1$

$I_7:$ the function is continuous

$I_8:$ the relation $\leq$ can be defined by π, i.e., $x \leq y \leftrightarrow \pi(x,y) = 1$

$I_9:$ $\pi(\pi(x,0),0) = x$

$I_{10}:$ $\pi(x,y) = \pi(1-y,1-x)$

for all $x, y, z \in [0,1]$. The properties $I_1^b, I_2, I_4, I_7, I_8$ and I_{10} are based on [13]. An operator defined by a function π that fulfills I_0 is called *implication* and if π additionally satisfies I_2 it is called a *strong implication* [20]. In particular in fuzzy control, however, often a t-norm is used to interpret the implication function. Table 2 shows which implication function has which properties. Note that only the ŁUKASIEWICZ implication function satisfies $I_0 - I_{10}$. In table 3 the property $\forall r, t \in [0,1] : \sup\{\kappa(\pi(r,s), \pi(s,t)) | s \in [0,1]\} = \pi(r,t)$, which implies κ-transitivity of π, is checked for different t-norms κ.

5 Conclusion

We presented general conditions for the chainability of fuzzy IF-THEN rules and analyzed various implication functions w.r.t. these conditions. The results enable the easy construction of fuzzy systems that can make use of the chaining syllogism, e.g., in order to speed up calculations in applications like expert or control systems.

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Name	Definition
Zadeh	$\pi_m(x,y) := \max(1-x, \min(x,y))$
Łukasiewicz	$\pi_a(x,y) := \min(1, 1-x+y)$
Mamdani (Minimum)	$\pi_c(x,y) := \min(x,y)$
Gaines-Rescher (Standard Strict)	$\pi_s(x,y) := \begin{cases} 1 \,;\, x \le y \\ 0 \,;\, \text{otherwise} \end{cases}$
Gödel-Heyting (Standard Star)	$\pi_g(x,y) := \begin{cases} 1 \,;\, x \le y \\ y \,;\, \text{otherwise} \end{cases}$
Standard Strict-Star	$\pi_{sg}(x,y) := \min(\pi_s(x,y), \pi_g(1-x, 1-y))$
Standard Star-Strict	$\pi_{gs}(x,y) := \min(\pi_g(x,y), \pi_s(1-x, 1-y))$
Standard Star-Star	$\pi_{gg}(x,y) := \min(\pi_g(x,y), \pi_g(1-x, 1-y))$
Standard Strict-Strict	$\pi_{ss}(x,y) := \min(\pi_s(x,y), \pi_s(1-x, 1-y)) = \mathrm{id}(x,y)$
Kleene-Dienes	$\pi_b(x,y) := \max(1-x, y)$
Gaines (Goguen)	$\pi_\Delta(x,y) := \begin{cases} 1 \quad ;\, x \le y \\ y/x \,;\, \text{otherwise} \end{cases}$
modified Gaines	$\pi_\blacktriangle(x,y) := \begin{cases} \min\{1, y/x, (1-x)/(1-y)\} & ;\, x > 0 \text{ and } y < 1 \\ 1 & ;\, \text{otherwise} \end{cases}$
Reichenbach (algebraic)	$\pi_*(x,y) := 1 - x + x \cdot y$
Willmott	$\pi_\#(x,y) := \min\{\max(1-x, y), \max(x, 1-x), \max(y, 1-y)\}$
Standard Sharp	$\pi_\Box(x,y) := \begin{cases} 1 \,;\, x < 1 \text{ oder } y = 1 \\ 0 \,;\, \text{otherwise} \end{cases}$
Wu 1	$\pi_{1b}(x,y) := \begin{cases} 1 & ;\, x \le y \\ \min(1-x, y) & ;\, \text{otherwise} \end{cases}$
Wu 2	$\pi_{1e}(x,y) := \begin{cases} 0 \,;\, x < y \\ y \,;\, \text{otherwise} \end{cases}$
Yager	$\pi_E(x,y) := y^x$ with $\pi_E(0,0) = 1$
weakly drastic	$\pi_{sd}(x,y) := \begin{cases} y & ;\, x = 1 \\ 1-x & ;\, y = 0 \\ 1 & ;\, \text{otherwise} \end{cases}$
drastic	$\pi_d(x,y) := \begin{cases} 0 \,;\, x = 1 \text{ and } y = 0 \\ 1 \,;\, \text{otherwise} \end{cases}$
probabilistic	$\pi_p(x,y) := x \cdot y$

Table 1: Implication functions, $x, y \in [0,1]$.

	I_0	I_1^a	I_1^b	I_1^c	I_2	I_3	I_4	I_5	I_6	I_7	I_8	I_9	I_{10}
π_m	✔	✔	✔							✔		✔	
π_a	✔	✔	✔	✔	✔	✔	✔	✔	✔	✔	✔	✔	✔
π_c			✔				✔			✔			
π_s	✔	✔		✔	✔			✔			✔		✔
π_g	✔	✔	✔	✔	✔		✔	✔	✔		✔		
π_{sg}								✔					
π_{gs}			✔				✔	✔					
π_{gg}			✔					✔					
π_{ss}					✔		✔	✔					✔
π_b	✔	✔	✔	✔	✔	✔	✔			✔		✔	✔
π_Δ	✔	✔	✔	✔	✔			✔	✔		✔		
$\pi_\blacktriangle$	✔	✔		✔	✔			✔			✔		✔
π_*	✔	✔	✔	✔	✔	✔	✔			✔		✔	✔
$\pi_\#$	✔		✔			✔	✔			✔		✔	✔
$\pi_\square$	✔	✔		✔	✔	✔	✔	✔	✔				
π_{1b}	✔	✔		✔	✔			✔			✔		✔
π_{1e}			✔				✔						
π_E	✔	✔	✔	✔	✔		✔						
π_{sd}	✔	✔	✔	✔	✔	✔	✔	✔	✔			✔	✔
π_d	✔	✔		✔	✔		✔	✔	✔				✔
π_p			✔				✔			✔			

Table 2: Properties of implication functions in table 1.

	$\min(x, y)$	$x \cdot y$	$\max(0, x + y - 1)$	$\min(x, y)$ if $\max(x, y) = 1$ and 0 otherwise
π_m				
π_a			✔	✔
π_c	✔			
π_s	✔	✔	✔	✔
π_g	✔	✔	✔	✔
π_{sg}	✔	✔	✔	✔
π_{gs}	✔	✔	✔	✔
π_{gg}	✔	✔	✔	✔
π_{ss}	✔	✔	✔	✔
π_b			✔	✔
π_Δ		✔	✔	✔
$\pi_\blacktriangle$		✔	✔	✔
π_*				
$\pi_\#$				
$\pi_\square$	✔	✔	✔	✔
π_{1b}	✔	✔	✔	✔
π_{1e}	✔			
π_E				
π_{sd}				
π_d				
π_p		✔		

Table 3: Presumption 1 of theorem 4 for $I = [0, 1]$, $Q = \sup$, four t-norms, and the implication functions in table 1.